ISRAEL JOURNAL OF MATHEMATICS 94 (1996), 247–271

EXTREME POINTS OF THE BLOCH SPACE OF A HOMOGENEOUS TREE

BY

JOEL M. COHEN

Department of Mathematics, University of Maryland
College Park, MD 20742, USA
e-mail: jmc@math.umd.edu

AND

FLAVIA COLONNA

Department of Mathematical Sciences, George Mason University
4400 University Drive, Fairfax, Virginia 22030, USA
e-mail: fcolonna@osf1.gmu.edu

ABSTRACT

In this work we study the Bloch space $\mathcal{B}$, the space of complex-valued harmonic functions f of a homogeneous tree with the property that $|f(u) - f(v)|$ is bounded for all pairs of neighboring vertices u and v. This space can be identified with a Besov–Lipschitz space of functions on the boundary of the tree. The unit ball $\mathcal{U}$ of $\mathcal{B}$ is convex and compact under the compact-open topology. Thus it is the closed convex hull of its extreme points. This paper gives necessary conditions and sufficient conditions for a harmonic function to be an extreme point. These conditions are not definitive in general, but they are for functions in the little Bloch space, that is, those satisfying the property that $|f(u) - f(v)| \to 0$, as u and v approach the boundary. This parallels the classical case precisely. In the real-valued case, we obtain a complete description. We also characterize the support points of $\mathcal{U}$.

1. Introduction

The classical theory of Bloch functions in the open unit disk and the interesting connections between the Bloch space and other function spaces was the motivation for the study [CC] of Bloch functions on trees. The article [CC] is part

Received February 9, 1993 and in revised form January 3, 1995

of a series of papers (e.g. [BCCP], [BFP], [CCP], [CP]) aimed at studying the remarkable relationship between the hyperbolic disk and infinite trees. In this article we continue this analysis, focusing on the study of the extreme points of the unit ball of the Bloch space, and considering also its support points.

We give a brief review of facts (cf. [ACP], [C2], [C3]) concerning the classical case which are relevant to our present work.

THEOREM 1: *Let f be analytic on Δ, the open unit disk. The following statements are equivalent:*

(1) *As a function between the metric spaces (Δ, ρ) and $(\mathbb{C}, d)$, f satisfies the Lipschitz condition with Lipschitz number $\beta_f = \sup_{z \neq w} \dfrac{d[f(z), f(w)]}{\rho(z,w)}$, where ρ is the hyperbolic distance and d is Euclidean distance.*
(2) *f is uniformly continuous.*
(3) *$\sup_{z\in\Delta}(1-|z|^2)|f'(z)|$ is finite and equals β_f.*
(4) *The set $\{f \circ S - f \circ S(0)$: S conformal automorphism of $\Delta\}$ constitutes a normal family.*
(5) *The schlicht disks contained in the image of f have bounded radius.*

We shall use the tree analogue of (1) as the definition of Bloch function on a tree. The functions of our study, however, are harmonic: there is not as natural a definition of analyticity on a tree. Bloch harmonic functions on the disk have also been studied (e.g. cf. [C1]).

We let $\mathcal{B}(\Delta)$ be the space of Bloch functions on Δ and let $\mathcal{B}_0(\Delta)$ be the subspace consisting of all Bloch functions f such that $(1-|z|^2)|f'(z)| \to 0$, as $|z| \to 1$, known as the **little Bloch space**.

An interesting classical result is the fact that $\mathcal{B}_0(\Delta)^{**}$ is isomorphic to $\mathcal{B}(\Delta)$. In [CC] we proved an analogous result for the tree case, using methods which are similar to those used in the classical case.

Another classical result is the identification of $\mathcal{B}(\Delta)$ with the Besov–Lipschitz space $B^0_{\infty,\infty}(\partial\Delta)$. In §2 (Theorem 3) we shall prove the tree analogue.

An interesting set of problems arises when one studies the extreme points of the unit ball of $\mathcal{B}(\Delta)$. These have been considered, for example, in [CW], [C3], and [B1]. One tool is the set of points $E(f) = \{z \in \Delta$: $(1-|z|^2)|f'(z)| = 1\}$. It is easy to see that f is an extreme point if $E(f)$ has limit points. (For example, $f(z) = 3\sqrt{3}z^2/4$ satisfies this condition.) For $f \in \mathcal{B}_0(\Delta)$, this is a necessary and sufficient condition. On the other hand, Bonk [B2] has proved that the Ahlfors–

Grunsky function [AG] is an extreme point for which $E(f)$ has no limit points. Similarly, in the present paper, in order to study the extreme points of the unit ball of the Bloch space of a homogeneous tree, we shall define the "skeleton" of a function on a tree, whose complement plays a role similar to that of $E(f)$. Once again, this set yields a condition which is sufficient (Theorem 6) but not necessary (Example 2). For the little Bloch space, however, it is both necessary and sufficient (Theorem 8).

Before giving the specific findings of this research, we recall the notation and the main results of previous works.

A **tree** is a connected and simply-connected graph $T = (V, E)$, where V is the set of vertices, and E is the set of edges, which are pairs of distinct vertices.

Two vertices v and w are called **neighbors** if there is an edge $e = [v, w]$, and we write $v \sim w$. Two edges e_1 and e_2 are called **neighbors** if $e_1 \cap e_2$ is a single vertex.

A **path** is a finite or infinite sequence of vertices $[v_0, v_1, \dots]$ such that $v_k \sim v_{k+1}$ and $v_{k-1} \neq v_{k+1}$, for all k. Equivalently, a path may be thought of as a sequence of neighboring edges $[e_0, e_1, \dots]$ (with $e_k = [v_k, v_{k+1}]$) such that $e_k \cap e_{k+1} \cap e_{k+2} = \emptyset$ for all k. An infinite path is also called a **ray**. A **geodesic** is the union of two rays meeting only at their initial vertex.

The condition for a graph to be a tree is equivalent to the following:

Between any two distinct vertices there exists a unique finite path

By abuse of terminology we will occasionally refer to a not-necessarily connected tree, by which we mean a disjoint union of (connected) trees. Some authors call this a **forest**, but we prefer not to introduce further terminology.

We define the **length** of a finite path $[v_0, \dots, v_n]$ to be n.

A vertex which has exactly two neighbors is called **flat**, and a path or ray is called **flat** if all its vertices, except possibly the first or last, are flat. A vertex which has exactly one neighbor is called **terminal**. We call an edge **terminal** if it contains a terminal vertex.

If p is the path $[v = v_0, v_1, \dots, v_n = w]$ or $[e_0, e_1, \dots, e_{n-1}]$, we say that the **distance $\mathbf{d(v, w)}$ from v to w** is n, the **distance $\mathbf{d(v, e_{n-1})}$ from v to $\mathbf{e_{n-1}}$** is $n-1$, and the **distance $\mathbf{d(e_0, e_{n-1})}$ from $\mathbf{e_0}$ to $\mathbf{e_{n-1}}$** is $n-1$.

Let u, v be neighbors. Then the **sector** determined by the ordered pair (u, v) is the set of all vertices w closer to v than to u, and the edges between them, hence, it is one of the two connected components of the complement of the edge

$[u, v]$.

If $d \geq 2$ is an integer, a tree is said to be **homogeneous of degree d** if each vertex has the same number d of neighbors, or equivalently, is in the same number d of edges.

Let $T = (V, E)$ be a tree. A **function on T** is a complex-valued function f on V. We use the notation $f: T \to \mathbb{C}$. In §3 we shall introduce functions on the directed edges as well, denoting these by $g: E \to \mathbb{C}$.

We recall the following definitions (cf. [CC]).

Definition 1: Let T be a homogeneous tree of degree d, with $d \geq 2$.

(1) A function $f: T \to \mathbb{C}$ is **harmonic** if for every vertex v of T, $f(v) = \frac{1}{d} \sum_{w \sim v} f(w)$.

(2) A function $f: T \to \mathbb{C}$ is **Bloch** if $\beta_f = \sup_{w \sim v} |f(w) - f(v)| < \infty$.

If T is homogeneous of degree 2, it consists of a single geodesic, and a harmonic function on T is necessarily constant, so any results are trivial. For this reason, we shall assume that the degree of T is greater than 2.

If f is a Bloch function, the number β_f, called the **Bloch constant** of f, is its Lipschitz number, where the function is thought of as a map between the metric spaces (T, d) and the Euclidean complex plane:

$$\beta_f = \sup_{v \neq w} \frac{|f(v) - f(w)|}{d(v, w)} = \sup_{v \sim w} |f(v) - f(w)|.$$

Let $\mathcal{B}$, the **Bloch space**, be the space of all Bloch harmonic functions on the homogeneous tree T. We define the **little Bloch space**, $\mathcal{B}_0$, to be the subspace of $\mathcal{B}$ consisting of all the functions f such that the set

$$\{v \in T \colon |f(v) - f(w)| \geq \epsilon, \text{ for some } w \sim v\}$$

is finite for all $\epsilon > 0$.

Fix a vertex o of T. We recall from [CC] that, as in the classical case, we have

THEOREM 2: *$\mathcal{B}$ is a complex Banach space under the norm defined by* $\|f\| = |f(o)| + \beta_f$.

Again, as in the classical case, the double dual of $\mathcal{B}_0$ is $\mathcal{B}$ ([CC]).

We shall study the unit ball of $\mathcal{B}$, with the aim of characterizing its extreme points. In §7 we completely characterize its support points.

Corresponding to a function in this unit ball, there is a certain subtree of the tree, called the skeleton, whose image lies in the open unit disk, and another set Λ, whose image lies in the unit circle.

We shall see (Theorem 6) that if the skeleton is empty, then the function is an extreme point, and in some cases (Corollary 1, Theorem 11, and Observation 2) — in particular if the function is real-valued — the converse holds. We shall also see (Theorem 12) that the function is a support point if and only if Λ is non-empty.

Other criteria such as the structure of the skeleton (Theorem 9) and growth conditions (Theorem 10) are explored for determining when a function is an extreme point.

We wish to express our deepest gratitude to Mitch Taibleson for patiently showing us how to relate the harmonic functions on the tree to functions on the boundary.

2. The Bloch space as a Besov–Lipschitz space

Before we begin the specifics of the study of the unit ball of the Bloch space, we wish to relate the Bloch space to a Besov–Lipschitz space of functions defined on the boundary of the tree. We adopt the point of view of [T2].

Let o be a fixed vertex of a homogeneous tree T of degree $d = q+1$. A **boundary point** of T is an equivalence class of rays any two of which differ by a finite number of vertices. In each class x there is a unique ray $[x_{-1}, x_0, x_1, \dots]$ starting at $x_{-1} = o$. We shall denote x as $\{x_{-1}, x_0, x_1, \dots\}$. Define the **boundary** of T as the set ∂T of boundary points of T.

A measure μ may be defined on ∂T as follows. Let $v \neq o$ be a vertex at distance $k+1$ from o. Then for $I_v = \{x \in \partial T\colon x_k = v\}$ let $\mu(I_v) = q^{-k}$. Since $I_o = \partial T$ is the disjoint union of the sets I_{v_j}, for $j = 1, \dots, q+1$, where $v_1, \dots, v_{q+1}$ are the neighbors of o, then $\mu(\partial T) = q+1$.

We put a partial ordering on $T \cup \partial T$ as follows. Let $v, w \in T$, $x \in \partial T$. Then $w \leq v$ if and only if $I_v \subset I_w$, and $w < x$ if and only if $x \in I_w$.

Let $\wedge$ be the symbol for the greatest lower bound, so that, for example, $w = w \wedge x$ if $x \in I_w$. For $v \in T$, let $|v| = d(o,v) - 1$, so that $\mu(I_v) = q^{-|v|}$, for $v \neq o$. Since $|o| = -1$, then $\mu(I_o) = q+1$, whereas $q^{-|o|} = q$.

Let u be an integrable function on ∂T with respect to μ. Then u induces a function φ on the tree by defining $\varphi(v) = \frac{1}{\mu(I_v)} \int_{I_v} u\, d\mu$. Then

$$(1) \qquad \varphi(v) = \frac{1}{q} \sum_{w \sim v, w \neq v^-} \varphi(w), \text{ for } v \neq o, \text{ and } \varphi(o) = \frac{1}{q+1} \sum_{w \sim o} \varphi(w),$$

where v^- denotes the (unique) neighbor of v closer to o. Since $I_v = \coprod_{w \sim v, w \neq v^-} I_w$, we get the first formula as follows:

$$\varphi(v) = q^{|v|} \sum_{w \sim v, w \neq v^-} \int_{I_w} u\, du = q^{|v|} \sum_{w \sim v, w \neq v^-} q^{-|v|-1} \varphi(w),$$

and the second formula is similar.

A function φ on T satisfying (1) is called a **martingale**.

Notice that the martingale property is very similar to harmonicity. In fact, there is a $1-1$ correspondence between the set of martingales $\{\varphi\}$ and the set of harmonic functions $\{f\}$ (cf. [T2]) in such a way that $\|f\|_\infty = \|\varphi\|_\infty$, where $\|h\|_\infty = \sup_{v \in T} |h(v)| \leq \infty$.

Given a martingale φ, we define the associated harmonic function f as follows. For $v \in T$, if $[x_{-1}, x_0, \ldots, x_n]$ is the path from o to v, then

$$f(v) = \frac{q-1}{q} \sum_{j=0}^{k} q^{-j} \varphi(x_{k-j}) + q^{-(k+1)} \varphi(x_{-1}).$$

Conversely, given a harmonic function f, the associated martingale φ is defined by $\varphi(o) = f(o)$, and

$$\varphi(v) = \frac{q}{q-1} \left(f(v) - \frac{1}{q} f(v^-) \right), \quad \text{for } v \neq o.$$

Let $\chi_v \colon \partial T \to \mathbb{C}$ be the characteristic function of I_v, and let $\mathcal{S}(\partial T)$ be the linear space of functions on ∂T generated by the set $\{\chi_v \colon v \in T\}$. A linear functional on $\mathcal{S}(\partial T)$ is called a **distribution**. Integrable functions h on ∂T as well as measures ν on ∂T induce distributions by letting $u(\chi_v)$ be $\int_{I_v} h d\mu$ or $\int_{I_v} d\nu$.

Given a distribution u, the associated function on T given by $\varphi(v) = u(\chi_v)/\mu(I_v)$ is a martingale. Conversely, a martingale φ determines a distribution u by $u(\chi_v) = \varphi(v)\mu(I_v)$. Therefore, there is also a $1-1$ correspondence between distributions and harmonic functions. This can be made explicit by means of the **Poisson kernel** function which is given by the formula

$$P(x,v) = \frac{q}{q+1} q^{2|x\wedge v|-|v|}.$$

Note that if $x \in I_v$, then

$$P(x,v) = \frac{q}{q+1} q^{|v|}.$$

The relationship between the distribution u and the corresponding harmonic function f can be given as the **Poisson integral**

$$f(v) = \int_{\partial T} P(x,v)u(x)d\mu(x).$$

Cf. [T2] for details. Taibleson proves that the same correspondence that holds for the classical case — the relationship between harmonic functions on the disk and distributions on its boundary — holds here.

Let f be a function on T. For $k = 0,1,2,\dots$ define the function f_k on ∂T by $f_k(x) = f(x_k)$, where $x = \{x_{-1}, x_0, x_1, \dots\}$, and let $d_k f = f_k - f_{k-1}$. With this notation, if φ is the martingale associated with the distribution u, then $u = \lim_{k\to\infty} \varphi_k$ (cf. [T2], (27)).

If v is a vertex with $|v| = k$, then $x_k = v$ for all $x \in I_v$. Since for all such v, $\mu(I_v) = q^{-k}$, we have

$$\|f_k\|_p = \left(\sum_{|v|=k} q^{-k} |f(v)|^p \right)^{1/p},$$

for $p > 0$, and we define $\|f\|_p = \sup_k \|f_k\|_p \le \infty$. Let $L^p(T) = \{f\colon \|f\|_p < \infty\}$. Letting $p \to \infty$, we see that $\|f_k\|_\infty = \max_{|v|=k} |f(v)|$ and $\|f\|_\infty = \sup_{v\in T} |f(v)|$. Observe that $\beta_f = \sup_k \|d_k f\|_\infty$, since if $v \sim w$, then $|f(v) - f(w)| = |d_k f(x)|$, for some k and for some $x \in \partial T$.

Let u be a distribution on ∂T and let f be its Poisson integral. A **cone** with vertex on ∂T can be defined (cf. [T2, §5]) yielding a notion of non-tangential limit. The following results can be found in [T2], (35–38):

(a) If u is an integrable function, then f converges to u non-tangentially a.e. and $\|u\|_\infty = \|f\|_\infty$.

(b) If u is continuous, then the sequence $\{f_k\}$ converges uniformly to u.

(c) If $1 \le p < \infty$ and $u \in L^p(\partial T)$, then $\{f_k\}$ converges to u in L^p.

(d) u is a Borel measure if and only if $f \in L^1(T)$.

(e) If $1 < p \leq \infty$, then $u \in L^p(\partial T)$ if and only if $f \in L^p(T)$.

PROPOSITION 1: *Let v and w be neighboring vertices. Then*

$$\int_{\partial T} |P(\xi, v) - P(\xi, w)| d\mu(\xi) = 2\frac{q-1}{q+1}.$$

Proof: Without loss of generality, we may let $w = v^-$, where $|v| = k$. Observe that $P(\xi, v) > P(\xi, w)$ if and only if $\xi \in I_v$. Thus

$$\begin{aligned}
&\int_{\partial T} |P(\xi, v) - P(\xi, w)| d\mu(\xi) \\
&= \int_{I_v} [P(\xi, v) - P(\xi, w)] d\mu(\xi) - \int_{\partial T - I_v} [P(\xi, v) - P(\xi, w)] d\mu(\xi) \\
&= 2\int_{I_v} [P(\xi, v) - P(\xi, w)] d\mu(\xi) - \int_{\partial T} [P(\xi, v) - P(\xi, w)] d\mu(\xi).
\end{aligned}$$

But $\int_{\partial T} P(\xi, -) d\mu(\xi)$ is the constant function 1, so the second integral vanishes. For $\xi \in I_v$ we have $P(\xi, v) = \frac{q}{q+1} q^k$ and $P(\xi, w) = \frac{q}{q+1} q^{k-1}$. Since $\mu(I_v) = q^{-k}$, this now yields

$$\int_{\partial T} |P(\xi, v) - P(\xi, w)| d\mu(\xi) = 2\frac{q}{q+1}(q^k - q^{k-1}) q^{-k} = \frac{2(q-1)}{q+1}. \quad \blacksquare$$

We now study Bloch functions on the tree in terms of sequences of functions on the boundary. First observe that from the definitions it follows that

$$d_k f(x) = \int_{\partial T} (P(\xi, x_k) - P(\xi, x_{k-1})) u(\xi) d\mu(\xi).$$

Since $\beta_f = \sup_k \|f_k\|_\infty$, the proposition implies that

$$\beta_f \leq \frac{2(q-1)}{q+1} \|u\|_\infty = \frac{2(q-1)}{q+1} \|f\|_\infty.$$

Furthermore, in order for equality to be attained, we must have a function u on ∂T such that $u|I_v = \lambda$ and $u|(\partial T - I_v) = -\lambda$, for some $\lambda \in \mathbb{C}$ and $v \in T$. This yields an alternate proof of Theorem 1 of [CC].

We now define the Besov–Lipschitz spaces.

Let u be a distribution on ∂T, $\alpha \geq 0$, $0 < p, s \leq \infty$, and let φ be the martingale associated to u. Then define

$$\|u\|_{p,s}^\alpha = \left(\sum_{k=0}^\infty \left(q^{k\alpha} \|d_k \varphi\|_{L^p(\partial T)} \right)^s \right)^{1/s},$$

with the appropriate modification for p or s infinite. Let $B^{\alpha}_{p,s}$ be the space of all distributions u such that $\|u\|^{\alpha}_{p,s} < \infty$. The norm on $B^{\alpha}_{p,s}$ is $\|u\|^{\alpha}_{p,s} + |f(o)|$. The identification of all the distributions which differ by a constant yields the space $\dot{B}^{\alpha}_{p,s}$ whose norm is just $\| \ \|^{\alpha}_{p,s}$. We now observe that with $\alpha = 0$ and $p = s = \infty$, the definition of $\|u\|^{\alpha}_{p,s}$ is precisely $\sup_{k,x} \|d_k\varphi(x)\|$. Now let f be the Poisson integral of u. Then the relation between f and φ yields

$$d_k f(x) = \frac{q-1}{q}\sum_{j=0}^{k} q^{-j} d_{k-j}\varphi(x)$$

and

$$d_k\varphi(x) = \begin{cases} \dfrac{q}{q-1}\left(d_k f(x) - \dfrac{1}{q} d_{k-1} f(x)\right) & \text{for } k = 1, 2, \ldots, \\ \dfrac{q}{q-1} d_0 f(x) & \text{for } k = 0. \end{cases}$$

This in turn yields

$$\beta_f \le \|u\|^{0}_{\infty,\infty} \le \frac{q+1}{q-1}\beta_f.$$

This proves the following

THEOREM 3: *As Banach spaces, the Bloch space $\mathcal{B}$ and and the Besov–Lipschitz space $B^{0}_{\infty,\infty}$ are isomorphic.*

We define a metric on ∂T by letting $d(x,y) = q^{-|x\wedge y|}$ for $x, y \in \partial T$, $x \neq y$. For $\alpha > 0$ we can show that $B^{\alpha}_{\infty,\infty}$ is the Lip-α space of $(\partial T, d)$. This is similar to Theorem (2.2), Ch. VI, in [T1].

THEOREM 4: *There exists $c > 0$ such that $u \in B^{\alpha}_{\infty,\infty}$ if and only if*

$$|u(x) - u(y)| \le c\|u\|^{\alpha}_{\infty,\infty} d(x,y)^{\alpha},$$

for all $x, y \in \partial T$.

Proof: Let φ be the martingale associated with u. Observe that $d_i\varphi(x) = d_i\varphi(y)$ for $i \le N = |x \wedge y|$. Thus

$$\varphi_k(x) - \varphi_k(y) = \sum_{i=0}^{k} [d_i\varphi(x) - d_i\varphi(y)] = \sum_{i=N+1}^{k} [d_i\varphi(x) - d_i\varphi(y)].$$

Since $\lim_{k\to\infty}\varphi_k = u$, we see that $u(x)-u(y)=\sum_{N+1}^{\infty}[d_i\varphi(x)-d_i\varphi(y)]$. Since $\|u\|_{\infty,\infty}^{\alpha}=\sup_{k,x}|d_k\varphi(x)|q^{k\alpha}$, we have that $|d_k\varphi(z)|\le\|u\|_{\infty,\infty}^{\alpha}q^{-k\alpha}$ for all $k\ge 0$ and all $z\in\partial T$, so that

$$|u(x)-u(y)|\le 2\sum_{i=N+1}^{\infty}\|u\|_{\infty,\infty}^{\alpha}q^{-i\alpha}=\frac{2q^{-N\alpha}}{q^{\alpha}-1}\|u\|_{\infty,\infty}^{\alpha}=\frac{2}{q^{\alpha}-1}\|u\|_{\infty,\infty}^{\alpha}d(x,y)^{\alpha}.$$

So we have proved that if $u\in B_{\infty,\infty}^{\alpha}$, then $|u(x)-u(y)|\le c\|u\|_{\infty,\infty}^{\alpha}d(x,y)^{\alpha}$, for all $x,y\in\partial T$, with $c=2/(q^{\alpha}-1)$.

On the other hand, if we assume that $|u(x)-u(y)|\le cd(x,y)^{\alpha}$ for some positive number c and all $x,y\in\partial T$, but that $\|u\|_{\infty,\infty}^{\alpha}=\infty$, then u is continuous, and for all $N\in\mathbb{N}$ there exists $x\in\partial T$ and $k\ge 0$ such that $|d_k\varphi(x)|q^{k\alpha}\ge N$. By the continuity of u we get

$$d_k\varphi(x)=q^k\int_{I_{x_k}}u-q^{k-1}\int_{I_{x_{k-1}}}u=u(z)-u(y),$$

for some $z\in I_{x_k}$ and $y\in I_{x_{k-1}}$. Since $I_{x_k}\subset I_{x_{k-1}}$, we have $d(z,y)\le q^{-(k-1)}$. Thus

$$N\le|d_k\varphi(x)|q^{k\alpha}\le|u(z)-u(y)|d(z,y)^{-\alpha}q^{\alpha}\le cq^{\alpha}.$$

Since N was arbitrary, this yields a contradiction. ∎

3. The unit ball of the Bloch space

Let $\mathcal{U}$ be the unit ball of the Bloch space, that is, the set of all harmonic functions f on T such that $|f(o)|+|f(v)-f(w)|\le 1$, for all pairs of neighboring vertices $v,w\in T$, where o is a fixed vertex of T. Clearly $\mathcal{U}$ is a convex set.

Beside the Banach space topology on $\mathcal{B}$, we may also consider the compact-open topology. It should be noted that this is the topology of uniform convergence on compact subsets. Since a compact set of vertices is finite, a sub-basis for this topology is the set of functions which send a given vertex into a given open set in $\mathbb{C}$. Clearly, then, convergence in this topology is equivalent to pointwise convergence. We next show that $\mathcal{U}$ is compact in this topology.

THEOREM 5: *$\mathcal{U}$ is compact in the compact-open topology.*

Proof: Since $\mathcal{U}$ is second countable, it is sufficient to show that it is sequentially compact. Let $\{f_n\}$ be a sequence of functions in $\mathcal{U}$. We can find a converging subsequence of $\{f_n\}$ and verify that its limit function f is in $\mathcal{U}$: for each vertex v, the set $\{f_n(v)\}$ is bounded, and thus has convergent subsequences. By induction on the distance of v from o and by a diagonalization process, we can find a subsequence which converges at all vertices. ∎

Thus by the Krein–Milman Theorem (cf. [R], p. 242) $\mathcal{U}$ is the closed convex hull of its extreme points. Our aim is to characterize the extreme points of $\mathcal{U}$. Although the problem is not completely resolved here — as it is also still unsolved in the classical case — we obtain many partial results.

Notice that the extreme points of $\mathcal{U}$ must have Bloch norm equal to one. Also f is an extreme point of $\mathcal{U}$ if and only if the condition $f = (g+h)/2$, for $g, h \in \mathcal{U}$, implies that $f = g = h$. Another way of expressing this same condition is that $f \pm k \in \mathcal{U}$, for k harmonic, implies that k is identically zero. This last statement will be the one most often used throughout this paper.

Since most of the study of a Bloch function on a homogeneous tree concerns the difference of its values at neighboring vertices, it is often convenient to look at a difference function defined on the edges. Since edges are unordered pairs, however, this difference is well-defined only up to sign. To overcome this difficulty we orient the edges. That is, if $e = [v, w]$ is an edge, we make a choice of one of its vertices to be the **initial vertex** $\iota(e)$ and the other to be the **terminal vertex** $\tau(e)$. We then call e a **directed edge**.

A function f on T then induces a function $g = f \circ \tau - f \circ \iota$ on the set of oriented edges E. Clearly f is Bloch if and only if the associated function g is bounded. Notice that f determines g and in turn g, together with the value of f at any one vertex, determines f. (The function g resembles a derivative of f in this respect.)

Given a vertex v, let W^+ be the set of all neighbors w of v which follow v. That is, if $\tau(e) = w$, then $\iota(e) = v$. Similarly, let W^- be the set of neighbors of v which precede v. Let f be a harmonic function on T. Since the sum of the cardinalities of W^+ and W^- is d, then $\sum_{w\in W^+}(f(w)-f(v)) + \sum_{w\in W^-}(f(w)-f(v)) = 0$. Thus

$$\sum_{\tau(e)=v} g(e) = \sum_{w\in W^+}(f(w)-f(v)) = \sum_{w\in W^-}(f(v)-f(w)) = \sum_{\iota(e)=v} g(e).$$

Hence $g = f\circ\tau - f\circ\iota$ satisfies Kirchhoff's Law

$$\sum_{\iota(e)=v} g(e) = \sum_{\tau(e)=v} g(e) \tag{2}$$

at each vertex v.

From now on we shall mainly consider functions on the directed edges of a homogeneous tree. There are two points that need to be addressed.

First we choose once and for all the orientation to be the one which is outward from o. That is, if $e = [v, w]$ and $d(o, v) < d(o, w)$, then $\iota(e) = v$ and $\tau(e) = w$. So for example for $v \neq o$, the set W^- above consists of the single vertex v^- (in the notation of §2), and for $v = o$, W^- is empty.

Note that $\|f\| = |f(o)| + \|g\|_\infty$, where $\| \ \|_\infty$ is the supremum norm for bounded functions of the edges. For simplicity of notation, in the future the subscript ∞ will be omitted.

The second point that needs to be made is that the map $f \mapsto g = f \circ \tau - f \circ \iota$ forgets the value $f(o)$. In the study of extreme points, shifting our focus from f to g, however, does not cause any loss of information, because of the following observation.

If f is an extreme point, then either f is a constant of modulus 1 or $f(o) = 0$: assuming $0 < |f(o)| < 1$ and $|f(o)| + \beta_f = 1$, and letting f_1 be the constant $\frac{f(o)}{|f(o)|}$ and $f_2 = (f - |f(o)|f_1)/\beta_f$, we obtain $f = |f(o)|f_1 + (1 - |f(o)|)f_2$ with $\|f_1\| = \|f_2\| = 1$, and f_1, f_2 distinct from f, so that f is not an extreme point.

On the other hand, if f is a constant of modulus 1 and h is any Bloch harmonic function such that $1 \geq \|f \pm h\| = |f(o) \pm h(o)| + \|h_1\|$, where $h_1 = h \circ \tau - h \circ \iota$, then from $|f(o)| = 1$ it follows that $h(o) = 0$ and $h_1 = 0$. Thus h is identically 0, so f is an extreme point.

Thus if $f \in \mathcal{B}$ is an extreme point of the unit ball, since f is either a constant of modulus 1 or maps o to 0, setting aside the first case, we may ask only about extreme points of the unit ball of the space $\{f \in \mathcal{B} | f(o) = 0\}$.

We call a function $g \colon E \to \mathbb{C}$ **harmonic** if g satisfies Kirchhoff's Law (2). Given such a function g, define a harmonic function f on the set of vertices as follows: For any vertex v, let $[e_1, \ldots, e_n]$ be the path from o to v. Define $f(v) = \sum_{k=1}^n g(e_k)$, so that $f(o) = 0$. Thus $g = f \circ \tau - f \circ \iota$. This shows that the set $\{f \in \mathcal{B} \colon f(o) = 0\}$ is in isometric 1-1 correspondence with the set $\mathcal{B}(E)$ of bounded harmonic functions on E under the map $f \mapsto g = f \circ \tau - f \circ \iota$. Under this correspondence, the unit ball $\mathcal{U}$ of $\mathcal{B}$ with the normalization at o is identified with the set $\mathcal{B}_1 = \{g \colon E \to \mathbb{C} \ | g \text{ harmonic}, \|g\| \leq 1\}$. Thus we are now studying the following:

PROBLEM: *Find the extreme points of the compact convex set $\mathcal{B}_1$.*

An advantage of the shift of focus from f to g is that the space $\mathcal{B}_1$ is independent of the specific choice of the fixed vertex o. Of course, a change in o determines a change in the orientation of a finite number of edges, and only changes the sign of the value of g on these edges.

4. The skeleton of an extreme point

Definition 2: If W is an arbitrary tree, a vertex v is said to be **hidden in W** if there are no (doubly infinite) geodesics of W which pass through v. An edge is **hidden in W** if one of its vertices is hidden.

Examples of hidden vertices are terminal vertices and flat neighbors of terminal vertices. Notice that an edge e is hidden in W if and only if it is hidden in the connected component C of W containing e. Furthermore e is hidden if and only if $C - \{e\}$ has at most one infinite component.

We shall denote by $W^{\#}$ the not-necessarily connected subtree consisting of all non-hidden vertices and non-hidden edges. Note that $W^{\#\#} = W^{\#}$.

Fix a function $g \in \mathcal{B}_1$ of supremum norm 1. Recalling that Δ is the open unit disk, consider the subtree $g^{-1}(\Delta)$ of T (possibly not connected), that is, the set consisting of those edges e such that $|g(e)| < 1$, together with their vertices.

Definition 3: Let $S(g) = (g^{-1}(\Delta))^{\#}$, the subtree (possibly disconnected) of $g^{-1}(\Delta)$ consisting of the non-hidden vertices and the non-hidden edges. We call $S(g)$ the **skeleton** of the function g. If f is the harmonic function on T such that $g = f \circ \tau - f \circ \iota$, the set $S(g)$ is also called the **skeleton** of f.

Let $g \in \mathcal{B}_1$. If e is an edge, we say that $g(e)$ is **determined** if whenever $g = (g_1 + g_2)/2$, with $g_1, g_2 \in \mathcal{B}_1$, it follows that $g(e) = g_1(e) = g_2(e)$. If X is a set of edges, we say that $g(X)$ is **determined** if $g(e)$ is determined for all $e \in X$.

The function g is harmonic on T, of course, but note that it is not generally harmonic on $S(g)$, as some of the edges may be missing.

Let $U(g)$ be the subtree (possibly not connected) consisting of the non-determined edges together with their vertices. Notice that g is an extreme point if and only if $U(g)$ is empty.

The following result is straightforward.

LEMMA 1: *If $|g(e)| = 1$ then $g(e)$ is determined.*

LEMMA 2: *Given a vertex v, let $e_1, \dots, e_d$ be all the edges which have v as vertex. If $g(e_j)$ is determined for all $j = 1, \dots, d-1$, then $g(e_d)$ is determined.*

Proof: Let $g = (g_1 + g_2)/2$, with $g_1, g_2 \in \mathcal{B}_1$. Then we have $g(e_j) = g_1(e_j) = g_2(e_j)$, for $j = 1, \dots, d-1$. Since g, g_1, g_2 are harmonic, by Kirchhoff's Law it follows that $g(e_d) = g_1(e_d) = g_2(e_d)$. ∎

LEMMA 3: *$U(g) = U(g)^{\#}$ and $U(g) \subset S(g)$.*

Proof: By Lemma 2, $U(g)$ cannot contain a terminal edge. Thus if e is any edge of $U(g)$ in the connected component C, then $C - \{e\}$ must be disconnected, but since $U(g)$ contains no terminal vertices, neither component of $C - \{e\}$ can be finite. Hence e is not hidden. Thus $U(g) = U(g)^{\#}$.

By Lemma 1 we obtain that $U(g) \subset g^{-1}(\Delta)$, so $U(g) = U(g)^{\#} \subset (g^{-1}(\Delta))^{\#} = S(g)$. ∎

It follows immediately that

THEOREM 6: *If the skeleton of g is empty, then g is an extreme point.*

We shall see in Corollary 1, Theorem 11, and in Observation 2 that with some further conditions on g, the converse to Theorem 6 also holds.

THEOREM 7: *Given $g \in \mathcal{B}_1$, if there is a geodesic γ such that $|g|$ is bounded away from 1 on γ, then g is not an extreme point.*

Proof: By hypothesis, we may assume that there exists $\epsilon > 0$ such that $|g(e)| \leq 1 - \epsilon$ for all $e \in \gamma$. Let $\gamma = \{e_n\}_{-\infty}^{\infty}$. Label the edges of γ so that e_{-1}, e_0 are the edges closest to o. Define the function h on E by setting $h(e) = 0$ for $e \notin \gamma$, $h(e_n) = \epsilon$ for $n \geq 0$, and $h(e_n) = -\epsilon$ for $n < 0$. Then h is harmonic and $g \pm h \in \mathcal{B}_1$. So g is not an extreme point. ∎

COROLLARY 1: *Let $g \in \mathcal{B}_1$. If the set $|g(g^{-1}(\Delta))|$ is bounded away from 1 (for example, if the image of $|g|$ is finite), then g is an extreme point if and only if $S(g)$ is empty.*

Proof: If $S(g)$ is empty then g is an extreme point by Theorem 6. Conversely, if $S(g)$ is non-empty, then since $S(g) = S(g)^{\#}$ contains no hidden edges, it must contain a geodesic γ. Since $S(g) \subset g^{-1}(\Delta)$, by the hypothesis $|g|$ is bounded away from 1 on $S(g)$, hence on γ. So by Theorem 7, g is not an extreme point. ∎

As an immediate consequence of Corollary 1, we note that if $g \in \mathcal{B}_1$ is a function which vanishes at infinity, then g is an extreme point if and only if $S(g)$ is empty. But for $f \in \mathcal{B}_0$, the little Bloch space, the corresponding function g vanishes at infinity. Thus we have

THEOREM 8: *Let $f \in \mathcal{B}_0$. Then f is an extreme point if and only if its skeleton is empty.*

Thus for functions in the unit ball of the little Bloch space the skeleton completely determines whether a function is an extreme point, just as in the classical case the set $E(f)$ is determinant for $f \in \mathcal{B}_0(\Delta)$.

LEMMA 4: *Let $\rho = \{e_n\}_1^\infty$ be a flat ray in the skeleton of a function $g \in \mathcal{B}_1$, and assume that $\sup_n |g(e_n)| = 1$. Then $g(\rho)$ is determined.*

Proof: Assume $g \pm h \in \mathcal{B}_1$, for some harmonic function h. Since by Lemma 1 g is determined off its skeleton, h is identically 0 there. By the harmonicity of h and the flatness of ρ, $h(e_n) = \pm h(e_1)$ for all $n \in \mathbb{N}$. Since $g \pm h \in \mathcal{B}_1$, this yields $|g(e_n) \pm h(e_1)| \leq 1$. Thus $|h(e_1)|^2 \leq 1 - |g(e_n)|^2$, for all $n \in \mathbb{N}$, which forces $h(e_1)$ to vanish. Thus h is identically 0 on ρ, and hence $g(\rho)$ is determined. ∎

THEOREM 9: *Assume that each component of $S(g)$ is a finite union of geodesics. Then g is an extreme point if and only if $\sup_{e\in\gamma} |g(e)| = 1$, for every geodesic $\gamma \in S(g)$.*

Proof: From Theorem 7 it follows that if g is an extreme point, then for every geodesic $\gamma \in S(g)$, $|g(\gamma)|$ cannot be bounded away from 1.

Conversely assume that $\sup_{e\in\gamma} |g(e)| = 1$ for every geodesic γ of $S(g)$. Let C be a component of $S(g)$. Let $U_C = U(g) \cap C$. By the hypothesis, C may be written as $\bigcup_{j=1}^k \rho_j \cup \{e_1, \ldots, e_m\}$, where $k \geq 2$, the ρ_j are flat rays, and the e_j are edges. If there are at least two rays ρ_i, ρ_j such that $|g(\rho_i)|$ and $|g(\rho_j)|$ are bounded away from 1, then there is a geodesic γ which is the union of ρ_i, ρ_j and a finite set of edges, and so $|g(\gamma)|$ is bounded away from 1, contradicting our assumption. Thus without loss of generality we may assume that $\sup |g(\rho_j)| = 1$ for each $j = 2, \ldots, k$. In particular, then, $g(\rho_j)$ is determined for $j = 2, \ldots, k$, by Lemma 4. Thus U_C is contained in $\rho_1 \cup \{e_1, \ldots, e_m\}$. By Theorem 6, $U(g) = U(g)^\#$ and so $U_C = U_C^\#$. But since $(\rho_1 \cup \{e_1, \ldots, e_m\})^\# = \emptyset$, we get $U_C = \emptyset$. But $U(g)$ is the union of the sets U_C over all components C of $S(g)$. Thus $U(g)$ is empty and so g is an extreme point. ∎

We now show that the finiteness hypothesis on $S(g)$ of Theorem 9 cannot be dropped.

Example 1: Let T be a homogeneous tree of degree 3. We shall construct a harmonic function g on the edges of T, such that if $d(e,o) = n$, then $|g(e)| = 1 - 2^{-n}$. Notice that this implies that along any geodesic γ, the function $|g|$ is not bounded away from 1. We shall then construct a non-zero harmonic function h such that if $d(e,o) = n$, then $|h(e)| \leq 2^{-n}$. Thus $g \pm h \in \mathcal{B}_1$, showing that g is not an extreme point.

We shall define $g(e)$ by induction on $d(e,o)$. Set $g(e) = 0$ for each edge e at o. Now assume that, for some positive integer k, $g(e)$ has been defined and $|g(e)| = 1 - 2^{-d(e,o)}$ whenever $d(e,o) < k$. Let e_1, e_2 be the two edges at distance k from o adjacent to some edge e at distance $k-1$. We define $g(e_1)$, $g(e_2)$ applying Observation 1 below to $\alpha = g(e)$ and $r = (1 + |\alpha|)/2$.

To define the function h, choose one of the edges e_0 at o and define

$$h(e) = \epsilon 2^{-1-d(e,e_0)}, \text{ where } \epsilon = \begin{cases} 1 & \text{if } d(e,e_0) = d(e,o), \\ -1 & \text{otherwise.} \end{cases}$$

Both g and h are harmonic and have the prescribed growth conditions.

OBSERVATION 1: *Given a complex number α and $r \geq |\alpha|/2$, there exist complex numbers β_1, β_2 of modulus r whose sum is α.*

We see this by letting β_1, β_2 be $r, -r$ if $\alpha = 0$, and otherwise be equal to $\frac{\alpha}{|\alpha|} r e^{\pm i\theta}$, where $\cos\theta = \frac{|\alpha|}{2r}$.

The following proposition allows us to analyze the conditions necessary for a function g to be an extreme point, by throwing away flat rays from the skeleton of g.

PROPOSITION 2: *Let ρ be a flat ray in $S(g)$. Then g is an extreme point if and only if $g(S(g) - \rho)$ is determined.*

Proof: To prove the proposition, we only need to show that if $g(e)$ is determined for each $e \in S(g) - \rho$ then $g(\rho)$ is determined. Let v_0 be the starting vertex of ρ. If e_1 is the edge of ρ through v_0, then by our hypothesis, the definition of $S(g)$, and Lemma 1, we have that for every edge e' of T through v_0 distinct from e, $g(e')$ is determined. Thus by Lemma 2 it follows that $g(e_1)$ is determined. Now arguing inductively, let e_n, $n \in \mathbb{N}$, be an edge of ρ such that $g(e_n)$ is determined.

Let e_{n+1} be the edge of ρ sharing a vertex v_n with e_n. Then g is determined at each edge at v_n except possibly e_{n+1}. Again by Lemma 2 we have that $g(e_{n+1})$ is determined. ∎

5. Growth conditions

Up to now we have seen two techniques for studying the main question. For $g \in \mathcal{B}_1$, if $S(g)$ is empty, then g is an extreme point. If the modulus of g is bounded away from 1 on a geodesic, then g is not an extreme point. In this section we shall consider functions g whose skeleton is the whole tree T (so that g is not necessarily an extreme point), but whose modulus is not bounded away from 1 on any geodesic (so that g is not necessarily a non-extreme point). We shall see by example that either possibility may occur.

Throughout this section we let T be a homogeneous tree of degree $d = q + 1$, with fixed vertex o.

Assume that g is a harmonic function on the edges of T whose modulus is everywhere less than 1, but approaches 1 along each ray. We would like to find growth conditions on $|g|$ for which g is necessarily an extreme point, or for which g is necessarily not an extreme point.

Example 1 shows how a growth condition can lead to such information in the case $d = 3$. More generally, still with $d = 3$, assume that g satisfies the condition that $|g(e)| \leq 1 - c2^{-n}$, $0 < c < 1$, for any edge e at distance n from o. The fact that g is not an extreme point will follow from Theorem 10(1).

In the other direction, we have the following example where g is an extreme point, yet $S(g)$ is the entire tree.

Example 2: Let $d = 3$. Let e_1, e_2, e_3 be the edges touching o. Define $g(e_t) = \frac{1}{2}e^{2t\pi i/3}$. On each of the sectors determined by these edges, use Observation 1 to define inductively $g(e)$ such that $|g(e)| = 1 - \frac{1}{2}5^{-n}$ for any edge e at distance n from o, and such that g is harmonic. Using Theorem 10(2) we will see that g is an extreme point, although $S(g)$ is the entire tree.

This shows that the growth of g along rays — and not just the supremum or the limit — may control whether or not g is an extreme point. This demonstrates the difficulty in getting a precise classification of the extreme points of the unit ball of the Bloch space.

PROPOSITION 3: *Let g be a harmonic function defined on the edges of T with $\|g\| \leq 1$, and let e_1 and e_2 be neighboring edges touching o. Let σ_1 and σ_2 be the sectors determined by e_1 and e_2. Suppose that for $j = 1, 2$ there exists a harmonic function $h_j: T \to \mathbb{C}$, not identically zero, such that $|g(e) \pm h_j(e)| \leq 1$, for each edge e in $T_j = \sigma_j \cup \{e_j\}$. Assume that $h_j(e_j)$ is real, $j = 1, 2$. Then g is not an extreme point.*

Proof: Without loss of generality we may assume that $h_1(e_1) \geq 0 \geq h_2(e_2)$ and that $h_1(e_1) \geq -h_2(e_2)$. Choose $\alpha \in (0, 1]$ so that $\alpha h_1(e_1) = -h_2(e_2)$. Then define the function h on the edges of T by $h|T_1 = \alpha h_1$, $h|T_2 = h_2$, and $h|T - (T_1 \cup T_2) = 0$. Observe that h is harmonic since $h(e_1) + h(e_2) = 0$, and that $|g \pm h| \leq 1$. Thus g is not an extreme point. ∎

In the proposition, the hypothesis $h_j(e_j)$ real may be replaced by the more general condition that there exist $c \in \mathbb{R}$ with $c \neq 0$ such that $ch_1(e_1) = h_2(e_2)$.

THEOREM 10: *Let g be harmonic on T, with $\|g\| \leq 1$. Denote by L the set of limit points of*

$$\left\{(1 - |g(e)|)^{1/d(o,e)}: e \in E\right\}.$$

(1) *If $\inf L > 1/q$, then g is not an extreme point.*
(2) *If $\sup L < 1/q^2$, then g is an extreme point.*

Proof: To prove (1), first observe that if $\alpha \in [0, 1]$ with $\alpha < \inf L$, then the set of edges $\{e \in E: (1 - |g(e)|)^{1/d(o,e)} \leq \alpha\}$ is finite. So assuming $1/q \leq \alpha < \inf L$, there exists a positive integer N such that whenever $d(o, e) \geq N$, we have $(1 - |g(e)|)^{1/d(o,e)} > \alpha$, that is, $|g(e)| < 1 - \alpha^{d(o,e)}$. Let T_1 be a branch of the tree rooted at o. Define the function H_1 on T_1 by $H_1(e) = q^{-d(o,e)}$. Then H_1 is a positive harmonic function on T_1 and $H_1(e) + |g(e)| \leq 1$, for all but finitely many e in T_1. Thus there exists a constant $c \in (0, 1)$ such that $cH_1(e) + |g(e)| \leq 1$, for all edges e. Setting $h_1 = cH_1$ we get that $|g \pm h_1| \leq 1$ on T_1. Extend h_1 to an arbitrary harmonic function on T. Now letting T_2 be a different branch of T rooted at o, we may construct a harmonic function h_2 on T_2 similarly, and apply Proposition 3 to see that g is not an extreme point.

To prove (2), assume $\sup L < 1/q^2$, and let α be a constant such that $\sup L < \alpha < 1/q^2$. Then there exists $N \in \mathbb{N}$ such that $(1 - |g(e)|)^{1/d(o,e)} \leq \alpha$, whenever $d(o, e) \geq N$. Let h be a harmonic function such that $|g \pm h| \leq 1$. Given any edge e_0, let $k = d(o, e_0)$. From the harmonicity of h we see that for each $n \in \mathbb{N}$

there exists an edge e_n such that $d(e_0, e_n) = n$, $d(o, e_n) = n + k$ and $|h(e_n)| \geq \frac{1}{q^n}|h(e_0)|$. Also $|g(e_n)| \geq 1 - \alpha^{n+k}$, for $n \geq N - k$. Now $|g(e) \pm h(e)| \leq 1$ implies $|g(e)|^2 + |h(e)|^2 \leq 1$. So $1 - 2\alpha^k\alpha^n + \alpha^{2n+2k} + \frac{1}{q^{2n}}|h(e_0)|^2 \leq 1$. Thus

$$|h(e_0)|^2 \leq q^{2n}(2\alpha^k\alpha^n - \alpha^{2n+2k}) \leq (2\alpha^k)(q^2\alpha)^n,$$

for all $n \geq N - k$. Since $q^2\alpha < 1$, this implies that $|h(e_0)| = 0$. Thus h vanishes identically and g is an extreme point. ∎

It is not clear what happens in the case of intermediate growth. The simplest case possible is that of a homogeneous tree of degree 3, and a harmonic function g satisfying the relation $|g(e)| = 1 - 3^{-d(o,e)}$. These conditions determine g up to an automorphism of the tree and multiplication by a complex constant of modulus 1. Unfortunately, we do not know if g is an extreme point. Thus we have a test case for which we do not know the answer.

6. The real-valued case

Throughout this section we shall deal with real-valued harmonic functions in $\mathcal{B}_1$ on a homogeneous tree T. We characterize those functions which are extreme points of $\mathcal{B}_1$. It turns out that they are exactly the functions whose range is contained in the set $\{-1, 0, 1\}$ and for which the skeleton is empty.

In the following example we show that there exists a real-valued harmonic function which goes to 1 in absolute value along every ray of a homogeneous tree of degree 3, but which is not an extreme point.

Example 3: Let e_1, e_2, e_3 be the edges touching o. We define a real harmonic function g recursively as follows. Let $g(e_1) = -1/2$ and $g(e_2) = g(e_3) = 1/4$. So g has been defined on a finite subtree which contains terminal edges, but with no flat vertices. Now assume that e is a terminal edge of the subtree in which g has already been defined. We shall extend g harmonically to both "children" of e and to both children of one of e's children. The tree of definition of the extended function has again terminal edges, but no flat vertices. The construction goes as follows: First assume that $g(e) = a > 0$. Let e', e'' be the two edges further out from e, and let e_1'', e_2'' be the two edges further out from e''. Then define $g(e') = \frac{1+a}{2}, g(e'') = -\frac{1-a}{2}, g(e_1'') = -\frac{3+a}{4}, g(e_2'') = \frac{1+3a}{4}$. In the case that $g(e) = -a < 0$, define all values to be the negatives of those given above. Where previously the absolute value at the terminal edge had been a, now the terminal edges have absolute values greater than or equal to $\frac{1+3a}{4}$, since $\frac{1+3a}{4} < \frac{1+a}{2} < \frac{3+a}{4}$.

Given any ray, let $\{e_n\}$ be a sequence of (not necessarily consecutive) edges on the ray each of which is terminal as in the above construction. This means skipping at most every other edge in the ray, and also that

$$|g(e_{n+1})| \geq \frac{1+3|g(e_n)|}{4}.$$

Consider the sequence defined recursively by $a_{n+1} = \frac{1+3a_n}{4}$ with $a_1 = \frac{1}{2}$. Then $a_n = 1 - \frac{1}{2}\left(\frac{3}{4}\right)^n$. Hence $|g(e_n)| \geq 1 - \frac{1}{2}\left(\frac{3}{4}\right)^n$. Thus $|g(e_n)| \to 1$, proving that $\sup_{e\in\rho}|g(e)| = 1$ for any ray ρ.

On the other hand, letting $\{e_n\}$ be the set of **all** edges of any ray from o, and noting that the sequence $b_1 = \frac{1}{2}$, $b_{n+1} = \frac{1+b_n}{2}$ satisfies the relation $b_{n+2} = \frac{3+b_n}{4}$, we see that $|g(e_n)| \leq b_n$. But $b_n = 1 - \frac{1}{2^n}$, so $|g(e_n)| \leq 1 - \frac{1}{2^n}$. Thus g is not an extreme point, by Theorem 10(1).

In order to prove the main result of this section, Theorem 11, we need some preliminary tools.

LEMMA 5: *Assume that k is a natural number, and $a, b_1, \ldots, b_{k+1} \in [-1,1]$ are such that $a = \sum_{j=1}^{k+1} b_j$. Let $\delta_j = \min\{|b_j|, 1-|b_j|\}$. Then $\sum_{j=1}^{k+1}\delta_j \geq \min\{|a|, 1-|a|\}$.*

Proof: In this proof we use the following notation: if $x \in \mathbb{R}$ let $\lfloor x\rfloor$ represent the greatest integer less than or equal to x, and let $\widehat{x} = x - \lfloor x\rfloor \in [0,1)$. Observe that if $x \geq 0$, then $|x| = x \geq \widehat{x}$, while if $x < 0$, then $|x| = -x = -(\lfloor x\rfloor+1)+1-\widehat{x} \geq 1-\widehat{x}$. Thus $|x| \geq \min\{\widehat{x}, 1-\widehat{x}\}$, for any $x \in \mathbb{R}$.

Note first that in the cases $a = 0, \pm 1$ the result is trivially true. Also, replacing each b_j and a by their negatives, if necessary, we may assume that $a > 0$. So we take $a \in (0,1)$.

Reindex the sequence $\{b_j\}$ so that

$$b_j \in \begin{cases} [-1,-\frac{1}{2}) \text{ whence } \delta_j = 1+b_j \text{ for } j = 1,\ldots,M_1, \\ [0,\frac{1}{2}) \text{ whence } \delta_j = b_j \text{ for } j = M_1+1,\ldots,M_2, \\ [-\frac{1}{2},0) \text{ whence } \delta_j = -b_j \text{ for } j = M_2+1,\ldots,M_3, \\ [\frac{1}{2},1] \text{ whence } \delta_j = 1-b_j \text{ for } j = M_3+1,\ldots,k+1. \end{cases}$$

Thus $a = A-B-N$ where $A = \sum_{j=1}^{M_2}\delta_j$, $B = \sum_{j>M_2}\delta_j$, and $N = M_1+M_3-k$. In particular, then, $N = \lfloor A-B\rfloor$ and $a = \widehat{A-B}$, so that $|A-B| \geq \min\{a, 1-a\}$. Hence

$$\sum_{j=1}^{k+1} \delta_j = A + B \geq |A - B| \geq \min\{a, 1-a\}.$$

This completes the proof. ∎

Corollary 2: *Assume that k is a natural number, and $a, b_1, \ldots, b_{k+1} \in [-1,1]$ are such that $a = \sum_{j=1}^{k+1} b_j$, and $0 \leq \epsilon \leq \min\{|a|, 1-|a|\}$. For $j = 1, \ldots, k$ define $\delta_j = \min\{\epsilon - (\delta_1 + \cdots + \delta_{j-1}), |b_j|, 1-|b_j|\}$. Then $\epsilon - (\delta_1 + \cdots + \delta_k) \leq \min\{|b_{k+1}|, 1-|b_{k+1}|\}$.*

Proof: We need to show that $\epsilon \leq \sum_{j=1}^{k} \delta_j + \min\{|b_{k+1}|, 1-|b_{k+1}|\}$. Observe that if, for any index j_0, we have $\delta_{j_0} = \epsilon - (\delta_1 + \cdots + \delta_{j_0-1})$, then $\delta_j = 0$ for all $j > j_0$ and the assertion is trivial. So we may assume that $\delta_j = \min\{|b_j|, 1-|b_j|\}$ for all $j \in \{1, \ldots, k\}$. Thus it is sufficient to prove that $\min\{|a|, 1-|a|\} \leq \sum_{j=1}^{k+1} \delta_j$, defining $\delta_{k+1} = \min\{|b_{k+1}|, 1-|b_{k+1}|\}$. The assertion now follows at once from Lemma 5. ∎

Theorem 11: *Let T be a homogeneous tree of degree d and let $g \in \mathcal{B}_1$ be a real-valued function. Then g is an extreme point if and only if $S(g) = \emptyset$ and the image of g is contained in the set*

$$R = \begin{cases} \{-1, 0, 1\} & \text{for } d \text{ odd}, \\ \{-1, 1\} & \text{for } d \text{ even}. \end{cases}$$

Proof: Assume that the image of g is not contained in the set $\{-1, 0, 1\}$. Choose an edge e_* with $g(e_*) \neq 0, \pm 1$. By harmonicity there is an adjacent edge e_{**} such that $g(e_{**}) \neq 0, \pm 1$. Without loss of generality, we may assume that o is their common vertex. We shall use Proposition 3 to prove that g is not an extreme point. It is thus sufficient to define a harmonic function h_* on the sector σ_* determined by e_*, yielding by symmetry the definition of a harmonic function h_{**} on the sector determined by e_{**}.

Set $h_*(e_*) = \min\{|g(e_*)|, 1-|g(e_*)|\} \neq 0$. Recursively, assume that we have defined $h_*(e)$ such that $h_*(e) \leq \min\{|g(e)|, 1-|g(e)|\}$, for all $e \in \sigma_*$ at distance less than n from e_*, for some positive integer n. Let e_0 be an edge at distance $n-1$ from e_*, and let $e_1, e_2, \ldots, e_{d-1}$ be the neighbors of e_0 at distance n.

Set $k = d - 2$. We wish to construct $h_*(e_j)$ so that $h_*(e_j) \leq \min\{|g(e_j)|, 1-|g(e_j)|\}$, $j = 1, \ldots, k+1$, and (to make h_* harmonic) $h_*(e_0) = \sum_1^{k+1} h_*(e_j)$. Set $a = g(e_0), \epsilon = h_*(e_0)$, and $b_j = g(e_j), j = 1, \ldots, k+1$. Then setting $h_*(e_j) = \min\{\epsilon - (h_*(e_1)+\cdots+h_*(e_{j-1})), |b_j|, 1-|b_j|\}$ for all $j = 1, \ldots, k$, and $h_*(e_{k+1}) = \epsilon - (h_*(e_1) + \cdots + h_*(e_k)) \geq 0$, we obtain that $h_*(e_{k+1}) \leq \min\{|b_{k+1}|, 1 - |b_{k+1}|\}$, by Corollary 2.

By induction, we have now defined h_* harmonic on $\sigma_* \cup \{e_*\}$. We can extend h_* arbitrarily to a harmonic function on all of T. We have that $|g \pm h_*| \leq 1$ on $\sigma_* \cup \{e_*\}$. Similarly there is a harmonic function h_{**} on T with $|g \pm h_{**}| \leq 1$ on $\sigma_{**} \cup \{e_{**}\}$. By Proposition 3, g is not an extreme point.

Next assume that the image of g is contained in the set $\{-1, 0, 1\}$. Then g is an extreme point if and only if the skeleton of g is empty by Corollary 1.

Finally we need to observe that if d is even, then an extreme point cannot take on the value 0. Let $e_1, \ldots, e_d$ be all the edges at some vertex v. Since $\sum_{\tau(e_j)=v} g(e_j) = \sum_{\iota(e_j)=v} g(e_j)$, there must be an even number of vanishing $g(e_j)$. Thus if $g(e) = 0$, by induction we can construct a geodesic through e on which g is identically 0. So by Theorem 7, g is not an extreme point. ∎

It is straightforward to construct a real extreme point g satisfying the conditions of Theorem 11.

7. Support points and further remarks

Recall that, given a complex topological vector space X and a subset A of X, a point $x \in A$ is called a **support point** of A if there exists a continuous linear functional L such that $L|A$ is non-constant and $\operatorname{Re} L(x) = \max_{y \in A} \operatorname{Re} L(y)$. In the case of $X = \mathbb{C}$ and A a closed convex polygon (together with its interior), the support points are precisely the points of the boundary of the polygon, whereas the extreme points are its vertices. In general, however, an extreme point is not necessarily a support point. In this section we shall characterize the support points for $X = \mathcal{B}$ and $A = \mathcal{B}_1$, and, in fact, we shall see that there are extreme points which are not support points.

Let X be a locally convex space, and let A be a compact subset of X. Given a continuous linear functional L on X, the set of support points of A associated with L is a nonempty compact subset of A. Hence it has extreme points, by the Krein–Milman Theorem. Using the linearity of L it is easy to see that any extreme point of this subset of A is actually an extreme point of A. Since $\mathcal{B}$ is locally convex even under the compact-open topology, the set of support points of any continuous linear functional yields an extreme point.

In Theorem 12 we characterize the support points of $\mathcal{B}_1$. For a function g on E, let $\Lambda(g)$ be the set of edges e such that $|g(e)| = 1$. We now get the following theorem, which is a precise analog of the result of M. Bonk [B1] for Bloch functions on the unit disk.

THEOREM 12: *The support points of $\mathcal{B}_1$ are precisely those functions g such that $\Lambda(g)$ is non-empty.*

Proof: We shall use the sequence of transformations $\{\chi_n\}_{n\in\mathbb{N}}$, introduced in [CC]. For convenience, we modify its definition to make it a transformation on functions of edges, instead of functions on vertices.

Let E_n be the set of edges both of whose vertices have distance $\leq n$ from o. For $g \in \mathcal{B}$, define $\chi_n g$ as follows:

First set $\chi_n g|E_n = g|E_n$. Then, if $e \notin E_n$, let e' be the element of E_n closest to e, and set $\chi_n g(e) = q^{-d(e,e')}g(e')$. It is easy to see that

(1) $\chi_n g$ is harmonic,

(2) $(\chi_n g - g)\,|E_n = 0$,

(3) $\|\chi_n g\| = \|g|E_n\|$.

Now let L be any linear functional on $\mathcal{B}$ which is continuous with respect to the topology of pointwise convergence. Given $g \in \mathcal{B}$, let $\{r_n\}$ be an arbitrary sequence of complex numbers and define $g_n = r_n(\chi_n g - g)$. Notice that by (2) above, g_n approaches 0 pointwise. Thus $r_n(L\chi_n g - Lg) = Lg_n$ goes to 0. But in order for this to be true for any possible sequence $\{r_n\}$, it must be true that $(L\chi_n g - Lg)$ is 0 for all sufficiently large values of n. This proves the following:

LEMMA 6: *Let L be any linear functional on $\mathcal{B}$ which is continuous with respect to the topology of pointwise convergence, and let $g \in \mathcal{B}$. Then there exists some $N \in \mathbb{N}$ such that $Lg = L\chi_n g$ for all $n \geq N$.*

We may now proceed with the proof of the theorem. Assume first that $g \in \mathcal{B}_1$ has the property that $\Lambda(g) \neq \emptyset$. Choose $e \in \Lambda(g)$, and define $L\colon \mathcal{B} \to \mathbb{C}$ by $Lh = \overline{g(e)}h(e)$. Then for all $h \in \mathcal{B}_1$, we have $\operatorname{Re} Lh \leq 1$, but $Lg = 1$. So g is a support point of $\mathcal{B}_1$.

On the other hand, let $g \in \mathcal{B}_1$ be such that for all $e \in E$, $|g(e)| < 1$. Let L be any functional as in the statement of the lemma such that $\operatorname{Re} Lg > 0$. By Lemma 6, we have $Lg = L\chi_n g$ for all n sufficiently large. Now let $c = \|\chi_n g\|$ which, by (3) above, is $\|g|E_n\|$, a positive number less than 1 by our assumption, since E_n is a finite set. In particular, then, the function $g_1 = c^{-1}\chi_n g$ is an element of $\mathcal{B}_1$, and has the property that $\operatorname{Re} Lg_1 = c^{-1} \operatorname{Re} L\chi_n g = c^{-1} \operatorname{Re} Lg > \operatorname{Re} Lg$. Thus g cannot be a support point, completing the proof of the theorem. ∎

As a consequence of Theorem 12, we now see that Example 2 yields an extreme point which is not a support point.

Since the subspace W of real-valued functions in $\mathcal{B}_1$ is closed and convex, the extreme points that we found in Theorem 11 are precisely the extreme points of W. Thus by the Krein–Milman Theorem, every element of W is the limit of finite linear combinations of these extreme points. Since the values that these extreme points take on at each edge are $-1, 0, +1$, it is clear that each element of W has the form $\sum_0^\infty c_n g_n$ where each g_n is an extreme point of W and $\{c_n\}$ is a sequence of positive numbers summable to 1.

Let $g \in \mathcal{B}_1$. Then the functions $h_1 = \operatorname{Re} g$, $h_2 = \operatorname{Im} g$ are in W. Thus we can write $h_1 = \sum_0^\infty a_n g_n$ and $h_2 = \sum_0^\infty b_n g_n$, where some of the coefficients may be zero to allow for the same set of g_n. Letting $\alpha_n = a_n + ib_n$, we can write $g = \sum_0^\infty \alpha_n g_n$. This proves

PROPOSITION 4: *Let $g \in \mathcal{B}_1$. Then there exist real extreme points g_n of W (which are described in Theorem 11) and complex numbers α_n, with non-negative real and imaginary parts, such that $\sum_0^\infty \alpha_n = 1+i$ and such that $g = \sum_0^\infty \alpha_n g_n$.*

OBSERVATION 2: *Assume that $g \in \mathcal{B}_1$ can be described by $g = \sum_0^k \alpha_n g_n$ where the g_n are as above and $k \in \mathbb{N}$. Then the range of g is finite, since it is contained in the set of all sums $\sum_0^k \alpha_n \epsilon_n$ where ϵ_n is $1, 0$ or -1. In particular $|g|$ is bounded away from 1 on $S(g)$. So by Corollary 1, we see that g is an extreme point if and only if $S(g)$ is empty.*

It would be interesting to see what role the coefficients α_n play in the general case in determining whether or not a function g is an extreme point.

References

[AG] L. V. Ahlfors and H. Grunsky, *Über die Blochsche konstante,* Mathematische Zeitschrift **42** (1937), 671–673.

[ACP] J. M. Anderson, J. Clunie and C. Pommerenke, *On Bloch functions and normal functions,* Journal für die reine und angewandte Mathematik **270** (1974), 12–37.

[BCCP] C. A. Berenstein, E. Casadio Tarabusi, J. M. Cohen and M. A. Picardello, *Integral geometry on trees,* American Journal of Mathematics **113** (1991), 441–470.

[BFP] W. Betori, J. Faraut and M. Pagliacci, *An inversion formula for the Radon transform on trees,* Mathematische Zeitschrift **201** (1989), 327–337.

[B1] M. Bonk, *The support points of the unit ball in the Bloch space,* Journal of Functional Analysis **123** (1994), 318–335.

[B2] M. Bonk, *An extremal property of the Ahlfors–Grunsky function,* preprint.

[CCP] E. Casadio Tarabusi, J. M. Cohen and M. A. Picardello, *The horocyclic Radon transform on trees,* Israel Journal of Mathematics **78** (1992), 363–380.

[CW] J. A. Cima and W. R. Wogen, *Extreme points of the unit ball of the Bloch space* $\mathcal{B}_0$, The Michigan Mathematical Journal **25** (1978), 213–222.

[CC] J. M. Cohen and F. Colonna, *The Bloch space of a homogeneous tree,* Boletín de la Sociedad Matemática Mexicana **37** (1992), 63–82.

[CP] J. M. Cohen and M. A. Picardello, *The 2-circle and 2-disk problems on trees,* Israel Journal of Mathematics **64** (1988), 73–86.

[C1] F. Colonna, *The Bloch space of bounded harmonic mappings,* Indiana University Mathematics Journal **38** (1989), 829–840.

[C2] F. Colonna, *Bloch and normal functions and their relation,* Rendiconti del Circolo Matematico di Palermo (II) **XXXVIII** (1989), 161–180.

[C3] F. Colonna, *Extreme points of a convex set of Bloch functions,* Seminars in Complex Analysis and Geometry, EditEl, Rende, Italy, 1990, pp. 23–59.

[R] H. L. Royden, *Real Analysis,* 3rd ed., Macmillan, New York, 1988.

[T1] M. H. Taibleson, *Fourier Analysis on Local Fields,* Math. Notes 15, Princeton University Press, Princeton, 1975.

[T2] M. H. Taibleson, *Hardy spaces of harmonic functions on homogeneous isotropic trees,* Mathematische Nachrichten **133** (1987), 273–288.